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The solution of some diffusion problems by Fourier transform methods

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Abstract—Fourier transform methods are applied to a number of problems in which sources of heat move steadily over the surface of a fixed heat-conducting medium. The work extends known results for heating over semi-infinite surfaces to heating over finite patches. The method of solution is described and illustrated by some typical examples of practical interest.

INTRODUCTION

THE PROBLEMS considered in this paper are typical of ones that arise in various physical situations. They belong to a class of problems in which sources of heat move steadily over the surface of a fixed medium, or equivalently in which there is surface heat transfer at a fixed location, past which there is uniform motion of fluid or material. Such problems have been studied since the early 1900s because of their wide applications to metal treatments (e.g. welding), to contaminant dispersal and to geophysical phenomena such as aquifers. Reference may be made to Carslaw and Jaeger [1] for a review of earlier work. More recently Caflish and Keller [2], Levine [3] and Evans [4] have examined the problem of quench front propagation when hot bodies (slabs and cylinders) are immersed in cooling fluid.

Most of the previous investigations, however, concentrate on cases where the cooling is taken to occur over a semi-infinite length. In contrast, the problems considered here are such that the surface heat transfer occurs over a finite length. Thus this work extends the previous investigations to more realistic and useful forms.

We examine four specific problems:

(I) The determination of the thermal field in a steadily moving slab of finite thickness with a uniform surface heat input along a strip of finite width on the upper surface, lying perpendicular to the motion, the remainder of the upper surface and the lower surface of the slab being taken to be insulated. This problem was first considered by Rosenthal [5].

(II) The problem as in (I), but with linear radiation, or Newton cooling, instead of a uniform heat input. The semi-infinite analogue of this problem was considered by Caflish and Keller, and also Levine.

(III) The cylindrical version of (II), namely the cooling of a finite section of a steadily moving, cylindrical rod by Newton cooling. The cooling of a semi-infinite rod was considered by Evans [4]. This problem has relevance to the cooling of reactor rods.

(IV) Finally, the problem again as in (I) but with Newton cooling on the lower side of the slab, instead of it being insulated. This type of problem was suggested by Rosenthal, and is of interest in ship-repair welding.

In all of these problems the method of solution employs Fourier transforms in the longitudinal direction enabling the heat equation to be reduced to an ordinary differential equation, provided due consideration is given to the behaviour of the solution at large distances. In problems (I) and (IV) this enables an explicit solution to be derived for the inverse transform in the form of infinite series which are easily computed. The advantage of the method is that it provides the solution directly and avoids the necessity of first constructing an explicit source function as in Carslaw and Jaeger [1] and Rosenthal [5] and then distributing sources appropriately over the slab. Here the same result is obtained naturally through the Fourier transform technique. The method also has the advantage of showing when an explicit solution is not possible. This is the case in problems (II) and (III) which are extensions of the solutions given by, for example, Caflish and Keller [2], Levine [3] and Evans [4]. These authors use the Wiener–Hopf technique to obtain an explicit, though complicated, solution for the mixed boundary-value problems which arise. The present technique results in an integral equation over a finite region for the unknown temperature distribution on the surface of the slab. A Fourier expansion of the temperature reduces this to an infinite system of algebraic equations in the unknown Fourier coefficients which is easily computed by truncation.

FORMULATION OF PROBLEM (II)

In order to fix ideas we shall present detailed analysis of problem (II) and then outline the relevant changes for the other three problems.

Consider a slab of material occupying the space $|X| < \infty, 0 \leqslant Y \leqslant d$, moving uniformly in the positive X-direction with speed U. The surfaces of the slab are insulated except along the finite section $Y = d$, $0 \leqslant X \leqslant l$ where heat is transferred according to New-

NOMENCLATURE

a_m	$2\pi m/\beta$, $m \in \mathbb{Z}$
$\mathbf{b}, \mathbf{b}'$	column vectors defined by (29)
c_m	Fourier coefficients of $f(x)$
d	slab thickness; radius of rod in problem (III)
$f(x)$	surface temperature function defined by (25), (26)
$f_n(y)$	defined by (53)
$g(x)$	$\partial\phi/\partial y(x, 1)$, $0 < x < \beta$
h	surface heat transfer coefficient
k	thermal conductivity
l	length of surface heating element
p_m, q_m	real and imaginary parts of c_m
q	Qd/kT_i, problem (I); Qd/kT_a, problem (IV)
s_n	$(s^2+n^2\pi^2)^{1/2}$, problems (I) and (II); $(s^2+\lambda_n^2)^{1/2}$, problem (III); $(s^2+\nu_n^2)^{1/2}$, problem (IV)
s	Péclet number, $Ud/2k$; also denoted by Pe in the figures
x, y	dimensionless coordinates
B	Biot number, hd/k
C, D	coefficient matrices, defined by (29)
$G(\ )$	Green's function
I_0, I_1	modified Bessel functions
J_0, J_1	Bessel functions
Q	heat flux
T	dimensional temperature
U	slab speed
X, Y	dimensionless coordinates
$\mathbb{Z}$	set of integers.

Greek symbols

α	Fourier transform variables
α_j	defined by (30)
β	length ratio, l/d
γ	$(\alpha^2+s^2)^{1/2}$
ε_n	constants ($\varepsilon_0 = \frac{1}{2}$; $\varepsilon_n = 1, n \geqslant 1$)
ϕ	modified dimensionless temperature [see (10)]
κ	thermal diffusivity
λ_n	zeros of J_1
ν_n	positive roots of $\nu \tan \nu = B$
Φ	Fourier transform of ϕ.

Subscripts

a	ambient
f	final
i	initial.

ton's law of cooling. Thus, if $T(X, Y)$ denotes the steady-state temperature of the slab, we have

$$U\frac{\partial T}{\partial X} = \kappa\nabla^2 T \quad 0 < Y < d, \quad |X| < \infty \tag{1}$$

$$\frac{\partial T}{\partial Y} = 0 \quad Y = d, \quad X < 0 \quad \text{and} \quad X > l \tag{2}$$

$$-k\frac{\partial T}{\partial Y} = h(T-T_a) \quad Y = d, \quad 0 \leqslant |X| \leqslant l \tag{3}$$

$$\frac{\partial T}{\partial Y} = 0 \quad Y = 0, \quad |X| < \infty. \tag{4}$$

Here k and κ denote, respectively, the thermal conductivity and diffusivity of the material, T_a is the ambient temperature in the cooling atmosphere and h is a surface heat transfer coefficient. In addition

$$T \to T_i \quad \text{as} \quad X \to -\infty, \quad T \to T_f \quad \text{as} \quad X \to \infty \tag{5}$$

where T_i and T_f denote the initial and final temperatures. In the above ∇^2 denotes the two-dimensional Laplacian. On introducing

$$(x, y) = d^{-1}(X, Y), \quad \theta = (T-T_a)/(T_i-T_a)$$
$$\beta = l/d, \quad s = Ud/2\kappa, \quad B = hd/k$$

we obtain

$$2s\frac{\partial\theta}{\partial x} = \nabla^2\theta \tag{6}$$

$$\frac{\partial\theta}{\partial y} = 0 \quad y = 1, \quad x < 0, \quad x > \beta \tag{7}$$

$$\frac{\partial\theta}{\partial y} + B\theta = 0 \quad y = 1, \quad 0 \leqslant x \leqslant \beta \tag{8}$$

$$\frac{\partial\theta}{\partial y} = 0 \quad y = 0, \quad |x| < \infty \tag{9}$$

plus appropriate boundary conditions as $|x| \to \infty$; here B is the Biot number, $2s$ is the Péclet number and ∇^2 denotes the dimensionless Laplacian.

METHOD OF SOLUTION

The key to the solution is the initial transformation

$$\theta(x, y) = 1 + e^{sx}\phi(x, y) \tag{10}$$

which produces a boundary-value problem for ϕ well suited for the application of Fourier transforms in the

x-direction. It follows that ϕ satisfies

$$(\nabla^2-s^2)\phi = 0 \quad \text{in} \quad |x|<\infty, \quad 0<y<1 \quad (11)$$

$$\frac{\partial\phi}{\partial y} = 0 \quad y=1, \quad x<0, \quad x>\beta \quad (12)$$

$$\frac{\partial\phi}{\partial y} = -B(\phi+\mathrm{e}^{-sx}) \equiv g(x) \quad y=1, \quad 0\leqslant x\leqslant\beta \quad (13)$$

$$\frac{\partial\phi}{\partial y} = 0 \quad y=0, \quad |x|<\infty \quad (14)$$

$$\phi = o(\mathrm{e}^{-sx}) \quad \text{as} \quad x\to-\infty \quad (15)$$

$$\phi = O(\mathrm{e}^{-sx}) \quad \text{as} \quad x\to\infty. \quad (16)$$

A consideration of the behaviour as $x\to-\infty$ of possible solutions of (11) subject to insulation boundary conditions on $y=0,1$ shows that (15) may be replaced by

$$\phi = O(\mathrm{e}^{sx}) \quad \text{as} \quad x\to-\infty. \quad (15a)$$

Thus $\phi = O(\mathrm{e}^{-s|x|})$ as $|x|\to\infty$, which ensures that the Fourier transform

$$\Phi(\alpha,y) = \int_{-\infty}^{\infty}\phi(x,y)\exp(\mathrm{i}\alpha x)\,\mathrm{d}x \quad (17)$$

with inverse

$$\phi(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi(\alpha,y)\exp(-\mathrm{i}\alpha x)\,\mathrm{d}\alpha \quad (18)$$

exists in the strip $\mathscr{D}: |\mathrm{Im}\,\alpha| < s$ of the complex α-plane.

FOURIER TRANSFORM OF THE EQUATIONS AND BOUNDARY CONDITIONS

The transform of (11) is

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}y^2}-\gamma^2\Phi = 0, \quad \alpha\in\mathscr{D} \quad (19)$$

where $\gamma = (\alpha^2+s^2)^{1/2}$ such that $\gamma = \alpha$ when $s=0$. The solution of (19) satisfying the appropriate boundary conditions on $y=0$ and $y=1$ is

$$\Phi(\alpha,y) = \frac{\cosh\gamma y}{\gamma\sinh\gamma}\int_0^\beta \mathrm{e}^{\mathrm{i}\alpha x'}g(x')\,\mathrm{d}x' \quad (20)$$

and $\phi(x,y)$ follows from (18):

$$\phi(x,y) = \frac{1}{2\pi}\int_0^\beta g(x')\int_{-\infty}^{\infty}\frac{\cosh\gamma y}{\gamma\sinh\gamma}\mathrm{e}^{\mathrm{i}\alpha(x'-x)}\,\mathrm{d}\alpha\,\mathrm{d}x'. \quad (21)$$

On using (10) and (13), we find that

$$\theta(x,y) = 1-B\int_0^\beta\theta(x',1)G(x-x',y)\,\mathrm{d}x' \quad (22)$$

where

$$G(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}(\alpha+\mathrm{i}s)x}\cosh\gamma y}{\gamma\sinh\gamma}\,\mathrm{d}\alpha \quad (23)$$

$$= \mathrm{e}^{sx}\sum_{n=0}^{\infty}\frac{\varepsilon_n\mathrm{e}^{-s_n|x|}(-1)^n\cos n\pi y}{s_n} \quad (24)$$

with $\varepsilon_0 = 1/2$ and $\varepsilon_n = 1$, $n\geqslant 1$ and $s_n = (s^2+n^2\pi^2)^{1/2}$. The series representation (24) has been obtained by contour integration. Thus $\theta(x,y)$ may be calculated from (22) once we have determined $\theta(x,1)$, $0\leqslant x\leqslant\beta$. Letting

$$\theta(x,1) = \mathrm{e}^{sx}f(x), \quad 0\leqslant x\leqslant\beta \quad (25)$$

and evaluating (22) at $y=1$, we obtain an integral equation for $f(x)$:

$$f(x) = \mathrm{e}^{-sx}-B\int_0^\beta f(x')\sum_{n=0}^{\infty}\frac{\varepsilon_n}{s_n}\exp(-s_n|x-x'|)\,\mathrm{d}x' \quad (26)$$

where we have used (24).

SOLUTION OF THE INTEGRAL EQUATION

To solve (26) it is convenient to take a Fourier representation of $f(x)$, and generate an infinite set of equations for the Fourier coefficients. Thus, on writing

$$f(x) = \sum_{m=-\infty}^{\infty}c_m\exp(\mathrm{i}a_m x), \quad 0\leqslant x\leqslant\beta, \quad (27)$$

with $\bar{c}_{-m} = c_m$ and $a_m = 2\pi m/\beta$, and on using the orthogonality properties of $\{\exp(\mathrm{i}a_m x)|m\in\mathbb{Z}\}$ on $[0,\beta]$, we find that the Fourier coefficients satisfy

$$C\mathbf{p} = \mathbf{b}, \quad D\mathbf{q} = \mathbf{b}' \quad (28)$$

where

$$\left.\begin{aligned} C_{ij} &= 4B\varepsilon_i\sum_{n=0}^{\infty}\frac{\varepsilon_n(1-\mathrm{e}^{-\beta s_n})s_n}{(s_n^2+a_i^2)(s_n^2+a_j^2)}-\beta\alpha_j\delta_{ij} \\ D_{ij} &= 4B\sum_{n=0}^{\infty}\frac{\varepsilon_n(1-\mathrm{e}^{-\beta s_n})a_i a_j}{s_n(s_n^2+a_i^2)(s_n^2+a_j^2)}+\beta\alpha_j\delta_{ij} \\ b_i' &= -\frac{a_i(1-\mathrm{e}^{-s\beta})}{(s^2+a_i^2)}, b_i = -\frac{s(1-\mathrm{e}^{-s\beta})}{(s^2+a_i^2)}. \end{aligned}\right\} \quad (29)$$

Here there is no implied summation convention, p_m and q_m denote the real and imaginary parts of c_m, δ_{ij} denotes the Kronecker delta and

$$\alpha_j = 1+2B\sum_{n=0}^{\infty}\frac{\varepsilon_n}{s_n^2+a_j^2}, \quad \mathbf{p} = \begin{pmatrix}p_0\\p_1\\\vdots\end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix}q_0\\q_1\\\vdots\end{pmatrix}. \quad (30)$$

The matrix equations (28) may be truncated and solved numerically yielding approximations to $\mathbf{p}$ and $\mathbf{q}$, and hence c_m, $m=0,1,\ldots$. Having found c_m, $f(x)$ is known from (27) and hence $\theta(x, y)$ may be calculated from (22), using (25).

SOLUTION OF PROBLEM (II)

It is found that for $x < 0$:

$$\left.\begin{aligned}
\theta(x,y) &= 1+2B\,e^{sx}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\varepsilon_m\varepsilon_n}{s_n(s_n^2+a_m^2)} \\
&\quad\times\{s_np_m-a_mq_n\}\{e^{s_n(x-\beta)}-e^{s_nx}\} \\
&\quad\times(-1)^n\cos n\pi y. \\
&\text{For } 0\leqslant x\leqslant\beta: \\
\theta(x,y) &= 1+2B\,e^{sx}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\varepsilon_m\varepsilon_n}{s_n(s_n^2+a_m^2)} \\
&\quad\times\{[s_np_m+a_mq_n]\,e^{-s_nx}+[s_np_m-a_mq_n] \\
&\quad\times e^{s_n(x-\beta)}+s_n[q_m\sin(a_mx) \\
&\quad-p_m\cos(a_mx)]\}(-1)^n\cos n\pi y. \\
&\text{For } x>\beta: \\
\theta(x,y) &= 1+2B\,e^{sx}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\varepsilon_m\varepsilon_n}{s_n(s_n^2+a_m^2)} \\
&\quad\times\{s_np_m+a_mq_n\}\{e^{-s_nx}-e^{s_n(\beta-x)}\} \\
&\quad\times(-1)^n\cos n\pi y.
\end{aligned}\right\}\qquad(31)$$

In order to determine the temperature distribution in the finite analogue of the problem considered by Caflish and Keller, i.e. problem (II), the Biot number B, Péclet number $2s$ and aspect ratio β must first be specified, then equations (28) solved for p_m and q_m, and finally the expressions (31) evaluated.

We now consider the changes that are necessary in the above analysis to deal with the other problems.

PROBLEM (I)

Uniform surface heat input along a finite strip

The specification of this problem is as in (1)–(5), except that (3) is replaced by

$$k\frac{\partial T}{\partial Y} = Q \quad Y = d,\quad 0 \leqslant X \leqslant l \qquad (32)$$

where Q is the constant flux of heat transferred into the slab (if $Q > 0$). In consequence, (8) now becomes

$$\frac{\partial \theta}{\partial y} = q \quad y = 1,\quad 0 \leqslant x \leqslant \beta \qquad (33)$$

where $q = Qd/kT_i$,* and (13) becomes

$$\frac{\partial \phi}{\partial y} = q\,e^{-sx} \quad y = 1,\quad 0 \leqslant x \leqslant \beta. \qquad (34)$$

Thus $\phi(x, y)$ is again given by (21) but with $g(x) = q\,e^{-sx}$; (21) is easily evaluated and $\theta(x, y)$ determined.

Solution of problem (I)

For $x < 0$:

$$\left.\begin{aligned}
\theta(x,y) &= 1+q\sum_{n=0}^{\infty}\frac{\varepsilon_n(-1)^n e^{(s+s_n)x}}{s_n(s+s_n)} \\
&\quad\times\{1-e^{-(s+s_n)\beta}\}\cos n\pi y. \\
&\text{For } 0\leqslant x\leqslant\beta: \\
\theta(x,y) &= 1+q\left\{\frac{1}{2s}\left[x+\frac{1}{2s}+s\left(y^2-\frac{1}{3}\right)\right]\right. \\
&\quad+\sum_{n=1}^{\infty}\frac{(-1)^n e^{(s-s_n)x}}{s_n(s-s_n)}\cos n\pi y \\
&\quad\left.-\sum_{n=0}^{\infty}\frac{\varepsilon_n(-1)^n e^{(s+s_n)(x-\beta)}}{s_n(s+s_n)}\cos n\pi y\right\}. \\
&\text{For } x>\beta: \\
\theta(x,y) &= 1+q\left\{\frac{\beta}{2s}+\sum_{n=1}^{\infty}\frac{(-1)^n e^{(s-s_n)x}}{s_n(s-s_n)}\right. \\
&\quad\left.\times[1-e^{-(s-s_n)\beta}]\cos n\pi y\right\}.
\end{aligned}\right\}\qquad(35)$$

These results are the same as those of Rosenthal, except for a change in notation.

PROBLEM (III)

Newton cooling of a finite section of a moving cylinder

Here (1)–(5) is the correct specification providing that we re-interpret X and Y as axial and radial cylindrical polar coordinates, respectively, and we take

$$\nabla^2 \equiv \frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}+\frac{1}{Y}\frac{\partial}{\partial Y}.$$

The analysis also follows closely, except that Bessel functions naturally arise in this case, but the conclusions are the same regarding condition (15). The transform of the governing equation becomes

$$\frac{d^2\Phi}{dy^2}+\frac{1}{y}\frac{d\Phi}{dy}-\gamma^2\Phi = 0 \qquad (36)$$

which has a bounded solution, satisfying the surface conditions, given by

$$\Phi(\alpha, y) = \frac{I_0(\gamma y)}{\gamma I_1(\gamma)}\int_0^{\beta} e^{i\alpha x'}g(x')\,dx'. \qquad (37)$$

Thus,

$$\phi(x,y) = \frac{1}{2\pi}\int_0^{\beta} g(x')\int_{-\infty}^{\infty}\frac{I_0(\gamma y)}{\gamma I_1(\gamma)}e^{i\alpha(x'-x)}\,d\alpha\,dx' \qquad (38)$$

and $\theta(x, y)$ is given by (22), with $G(x, y)$ now defined

* Note that T_a is redundant in this case and may be set equal to zero.

by

$$G(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{I_0(\gamma y)}{\gamma I_1(y)} e^{-i(\alpha+1s)}\,d\alpha. \tag{39}$$

Again, by using contour integration, we may obtain a series representation for G:

$$G(x,y) = e^{sx}\sum_{n=0}^{\infty} \varepsilon_n \frac{e^{-s_n|x|} J_0(\lambda_n y)}{s_n J_0(\lambda_n)} \tag{40}$$

where λ_n, $n = 0,1,2,\ldots$ are the zeros of J_1 and $s_n = (s^2+\lambda_n^2)^{1/2}$. The analysis now goes through exactly as before and the results for $\theta(x,y)$ are given by (31) *except* that $(-1)^n \cos n\pi y$ must be replaced by $J_0(\lambda_n y)/J_0(\lambda_n)$ and s_n has the definition given above.

PROBLEM (IV)

Uniform surface heat input along one side of a strip with Newton cooling along the other side

In this case (3), (4) and (5) respectively, become

$$k\frac{\partial T}{\partial Y} = Q, \quad Y = d, \quad 0 \leqslant X \leqslant l \tag{41}$$

$$k\frac{\partial T}{\partial Y} = h(T-T_a)$$

$$Y = 0, \quad |X| < \infty \tag{42}$$

$$T \to T_a \quad \text{as} \quad |X| \to \infty. \tag{43}$$

In this case T_i is redundant and, for convenience, we set it equal to $2T_a$. Thus (8) and (9), respectively, become

$$\frac{\partial\theta}{\partial y} = q \quad y = 1, \quad 0 \leqslant x \leqslant \beta \tag{44}$$

with $q = Qd/kT_a$, and

$$\frac{\partial\theta}{\partial y} - B\theta = 0 \quad y = 0, \quad |x| < \infty \tag{45}$$

with $B = hd/k$. Furthermore (13) and (14) become

$$\frac{\partial\phi}{\partial y} = q e^{-sx} \quad y = 1, \quad 0 \leqslant x \leqslant \beta \tag{46}$$

$$\frac{\partial\phi}{\partial y} - B\phi = 0 \quad y = 0, \quad |x| < \infty \tag{47}$$

where now

$$\phi(x,y) = e^{-sx}\theta(x,y). \tag{48}$$

The solution of (19) satisfying the modified boundary conditions on $y = 0$ and $y = 1$ is

$$\Phi(\alpha,y) = \frac{q(\gamma\cosh\gamma y + \lambda\sinh\gamma y)}{\gamma(\gamma\sinh\gamma + \lambda\cosh\gamma)}\int_0^{\beta} e^{i(\alpha+is)x}\,dx. \tag{49}$$

Hence,

$$\theta(x,y) = q\int_0^{\infty} \tilde{G}(x-x',y)\,dx' \tag{50}$$

where

$$\tilde{G}(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(\gamma\cosh\gamma y + B\sinh\gamma y)}{\gamma(\gamma\sinh\gamma + B\cosh\gamma)} \times e^{-i(\alpha+is)x}\,d\alpha \tag{51}$$

$$= e^{sx}\sum_{n=0}^{\infty} \frac{e^{-s_n|x|}}{s_n} f_n(y) \tag{52}$$

with

$$f_n(y) = \frac{(v_n\cos v_n y + B\sin v_n y)}{(B+1)\sin v_n + v_n\cos v_n}. \tag{53}$$

Here v_n, $n = 1,2,\ldots$ are the positive roots of the transcendental equation $v\tan v = B$, and (52) has been determined by evaluating (51) by contour integration.

Again it is straightforward to determine $\theta(x, y)$ from (50) and (53).

Solution of problem (IV)

$$\theta(x,y) = \begin{cases} q e^{sx}\displaystyle\sum_{n=1}^{\infty} \frac{f_n(y)e^{s_n x}}{s_n(s+s_n)} \times [1-e^{-(s+s_n)\beta}], & x < 0 \\ q\displaystyle\sum_{n=1}^{\infty} f_n(y)\left\{\frac{2}{v_n^2} - \frac{1}{s_n}\left[\frac{e^{(s-s_n)x}}{s_n-s} + \frac{e^{(s+s_n)(x-\beta)}}{s_n+s}\right]\right\}, & 0 \leqslant x \leqslant \beta \\ q\displaystyle\sum_{n=1}^{\infty} \frac{f_n(y)e^{(s-s_n)x}}{s_n(s_n-s)} \times [e^{(s_n-s)\beta}-1], & x > \beta. \end{cases} \tag{54}$$

Here $s_n = (s^2+v_n^2)^{1/2}$.

RESULTS AND DISCUSSION

In each of the problems considered there are a number of different parameters involved. Rather than present an exhaustive study of the variation in temperature over a complete parameter range, we content ourselves with a few representative curves for each case which illustrate the more interesting features of the problem.

Problem (I) describes the distribution of temperature in a steadily moving slab of finite thickness where a uniform surface heat input is applied along a finite width strip on the upper surface. The results are given by equation (35) in terms of the dimensionless heat input $q = Qd/\kappa T_i$, the aspect ratio $\beta = l/d$ and the Péclet number $2s = Ud/\kappa$. The expressions given by (35) can be shown to agree with the work of Rosenthal [5] although his conclusion concerning the maximum temperature [5, p. 853] is at variance with

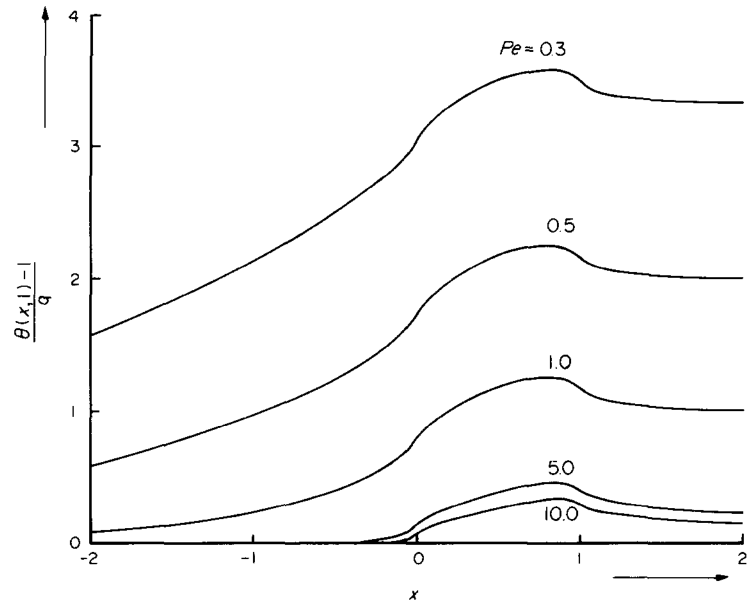

Fig. 1. Variation of the surface temperature around the section of the slab with uniform surface heat flux. The slab surface is insulated away from the heated section and $\beta = 1.0$.

ours. Thus Fig. 1 shows the variation of a normalised temperature on the upper surface $[\theta(x, 1)-1]/q$ with distance for the case $\beta = 1$ and for a range of increasing Péclet number. As expected, as $2s$ increases the heat is convected downstream more rapidly, preventing a build up of heat and resulting in a lower maximum for $\theta(x, 1)$. In all cases the maximum occurs at an interior point of the heated strip, in contrast to the suggestion of Rosenthal that it will always occur at the downstream end of the strip. This maximum shifts to the downstream end of the heated strip as $2s \to \infty$. The final downstream value of normalised temperature is $\beta/2s$ in agreement with a simple global conservation of energy calculation.

Figures 2 and 3 give contour plots of $(\theta-1)/q$ throughout the slab for two different values of Péclet

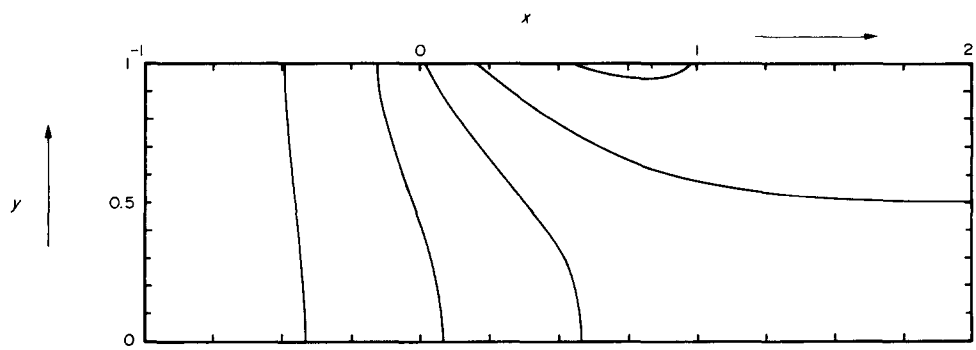

Fig. 2. Contour plots of $(\theta-1)/q$ for problem (I) with $Pe = 1.0$ and $\beta = 1.0$.

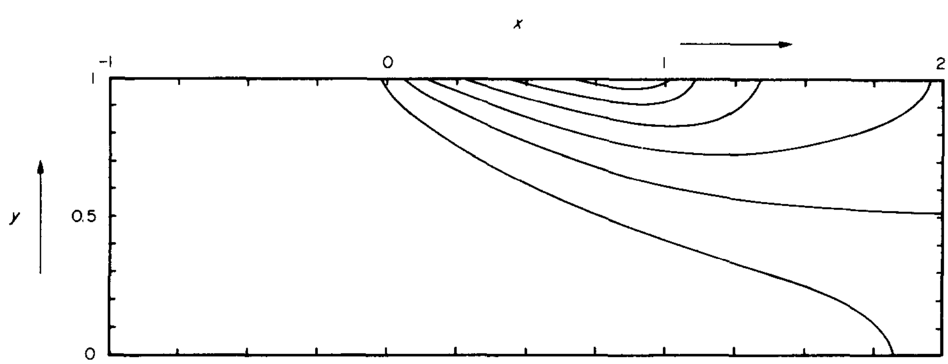

Fig. 3. Contour plots of $(\theta-1)/q$ for problem (I) with $Pe = 10.0$ and $\beta = 1.0$.

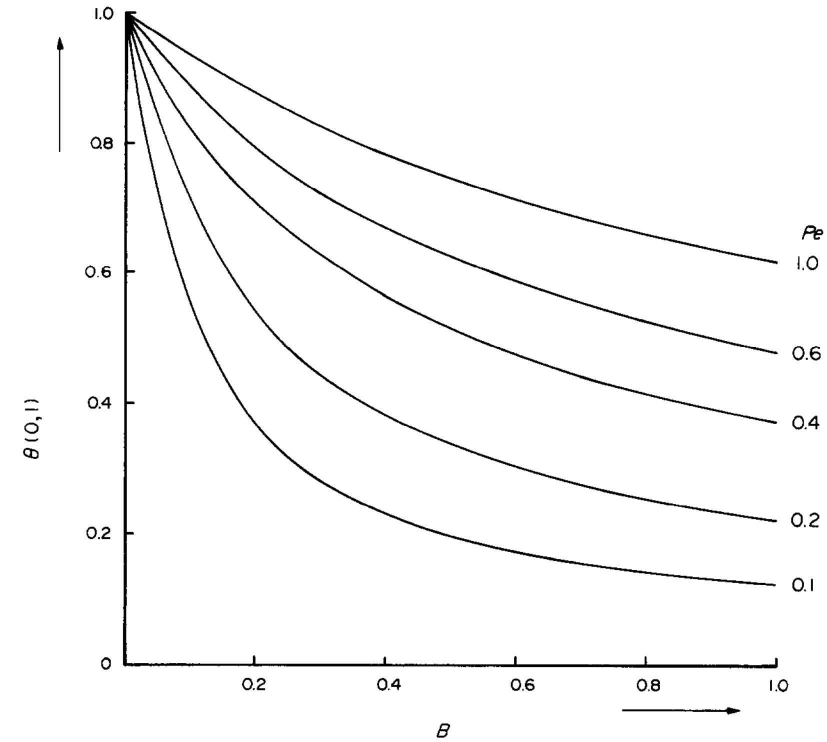

FIG. 4. Variation of the sputtering temperature of the slab with Biot number for $\beta = 1.0$ and various Péclet numbers—problem (II).

number. Again it is clear how the isotherms are distorted due to convective effects. In all contour plots no numbers are shown as we wish merely to illustrate qualitatively the distortion of the thermal field.

In Problem (II) the moving slab is being cooled over a finite strip by a linear radiation law, or Newton's law of cooling. The solution, given by equations (28) and (31) extends the results of Caflish and Keller [2] and Levine [3] where cooling takes place over a semi-infinite strip. Problem (III) is a simple extension to a moving circular cylinder rather than a slab and the solution, which follows similar lines to Problem (II), extends the solution given by Evans [4] for the semi-infinite case.

Results for Problem (II) are shown in Fig. 4 where the variation in what Levine calls the sputtering temperature, that is, the temperature on the upper side of the slab at the point of entry to the cooling region, is shown, as a function of Biot number, $B = hd/k$, for different values of $2s$, and for $\beta = 1$. It can be seen that the sputtering temperature falls monotonically as B increases, whilst for fixed B, it increases with the Péclet number. This is to be expected since as the speed of the slab increases there is less time for cooling to take place. The corresponding curves for Problem (III) are not shown as the differences between the two cases are so small. For example with the Péclet number based on the radius of the cylinder the differences in sputtering temperature in the two problems for the parameter ranges shown in Fig. 4 are of the order of 1%.

It is of interest to compare the results in Fig. 4 with

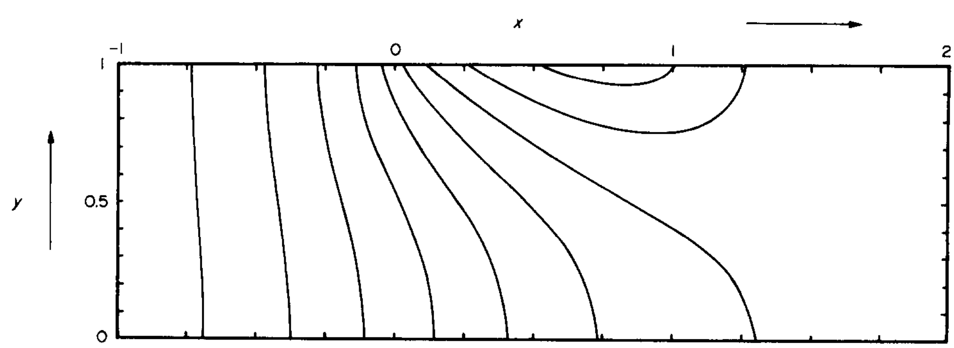

FIG. 5. Isotherms for $\beta = 1.0$, $B = 1.0$ and $Pe = 1.0$—problem (II).

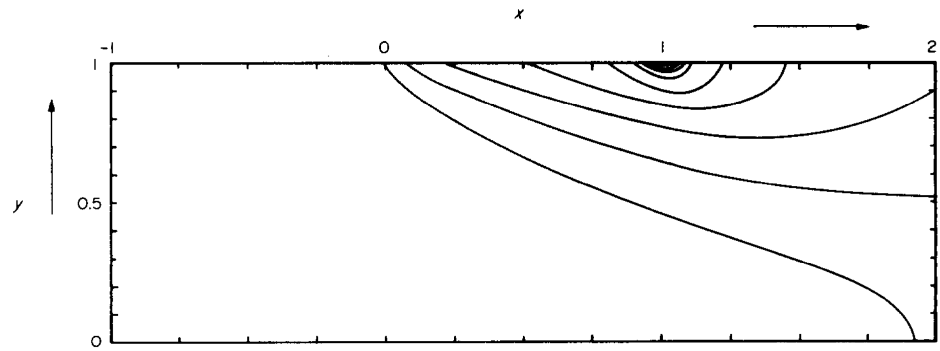

FIG. 6. Isotherms for $\beta = 1.0$, $B = 1.0$ and $Pe = 10.0$—Problem (II).

the curves given by Evans for the semi-infinite case. As expected each of the curves for given Péclet number $2s$ is higher than the corresponding curve for the semi-infinite case where more cooling takes place. Computations for increasing β confirm that the results tend to the semi-infinite case. Contour plots of constant temperature for the moving slab are shown in Figs. 5 and 6 for $2s = 1$ and 10, respectively. In each case $\beta = 1$ and the Biot number $B = 1$. In contrast to Problem (I) heat is now being *extracted* over the strip $0 < x < 1$ and it is clear that the minimum temperature occurs at an interior part of the cooled strip. For the larger value of $2s$ cooling is inhibited and the minimum shifts towards the downstream end of the strip.

The solution to Problem (IV) computed from (54) is illustrated in Figs. 7–9. Thus Fig. 7 shows how the temperature on the heated strip, normalised by q, varies with distance, again for fixed $\beta = 1$, $B = 1$ and for various values of Péclet number. Again the maximum temperature is attained at an interior point of the strip with the larger values corresponding to smaller Péclet number but downstream of the strip the situation is more complicated with the temperature curves overlapping. (See ref. [1, p. 270, Fig. 34] where a similar phenomenon occurs.) Contour plots of con-

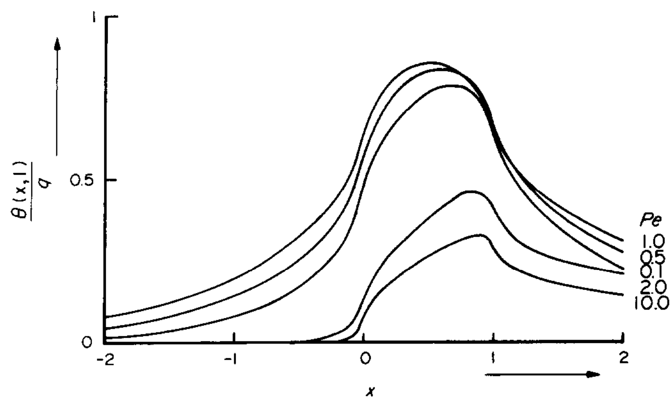

FIG. 7. Variation of the surface temperature around the section of the slab with uniform surface heat flux. The lower slab surface is cooled by linear radiation with $B = 1.0$ and $\beta = 1.0$.

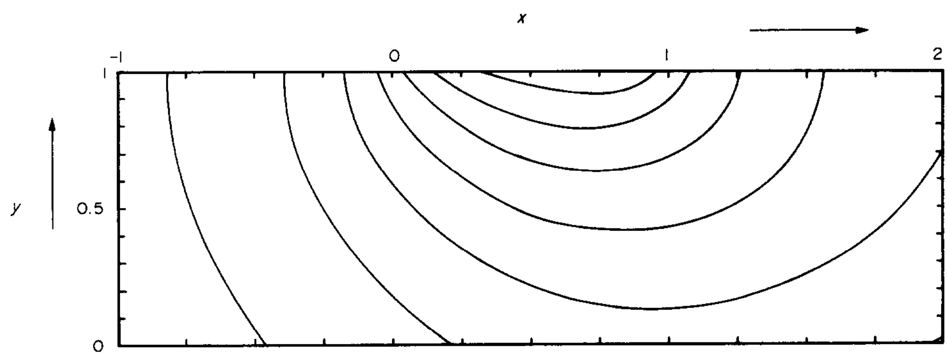

FIG. 8. Contour plots of θ/q for $\beta = 1.0$, $B = 1.0$ and $Pe = 1.0$—Problem (IV).

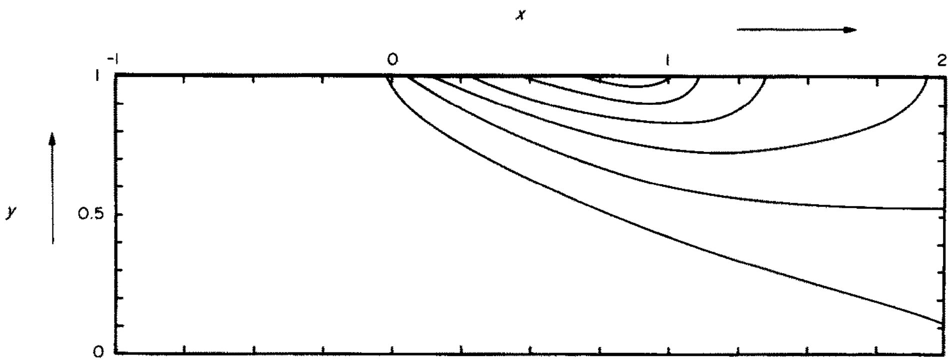

FIG. 9. Contour plots of θ/q for $\beta = 1.0$, $B = 1.0$ and $Pe = 10.0$—Problem (IV).

stant temperature are shown in Figs. 8 and 9 for values of $\beta = 1$, $B = 1$ and Péclet numbers of 1 and 10, respectively. It is clear from Fig. 8 that for $2s = 1$ the thermal distribution is almost symmetrical (as it would be if $2s = 0$) whereas from Fig. 9 where $2s = 10$ an appreciable amount of heat is being convected away.

CONCLUSION

A variety of problems concerning heating or cooling of moving materials have been considered, all of which have relevance to industrial applications. Fourier methods have been used to derive either, explicit solutions in the case of Neumann problems, or, for mixed boundary-value problems, a semi-explicit solution involving the numerical solution of an infinite system of linear equations. Representative graphs illustrating the main features of the solutions have been presented in each case, though the forms of the solutions do make it possible to consider more detailed aspects of the thermal distribution, if required.

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LA SOLUTION DE QUELQUES PROBLEMES DE DIFFUSION PAR LES METHODES DE TRANSFORMATION DE FOURIER

Résumé—Les méthodes de transformation de Fourier sont appliquées à un nombre de problèmes dans lesquels des sources de chaleur se déplacent à vitesse constante sur la surface d'un milieu fixe, conducteur de la chaleur. Le travail élargit des résultats connus pour le chauffage sur des réseaux finis. La méthode de résolution est décrite et illustrée par quelques exemples typiques d'intérêt pratique.

DIE LÖSUNG VON DIFFUSIONSPROBLEMEN MIT HILFE DER FOURIER-TRANSFORMATION

Zusammenfassung—Fourier-Transformationsverfahren werden auf eine Reihe von Problemen angewandt, in denen Wärmequellen stetig über die Oberfläche eines festen, wärmeleitenden Mediums wandern. Die Arbeit geht über bekannte Ergebnisse für die Beheizung von halbunendlichen Oberflächen hinaus und ermöglicht die Berechnung der Beheizung von endlichen Teilstücken. Das Lösungsverfahren wird beschrieben und anhand einiger typischer Beispiele, die von praktischem Insteresse sind, erläutert.

РЕШЕНИЕ НЕКОТОРЫХ ЗАДАЧ ДИФФУЗИИ С ПОМОЩЬЮ ПРЕОБРАЗОВАНИЯ ФУРЬЕ

Аннотация—Методы преобразования Фурье применимы к ряду задач, в которых источник тепла равномерно движется над поверхностью неподвижной теплопроводной среды. В работе известные результаты по нагреву сверху полубесконечных тел распространены на случай нагрева конечных областей. Метод решения описан и проиллюстрирован типичными примерами, имеющими практический интерес.